Some remarks on the projective properties of Menger and Hurewicz

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Here we are concerned with the case when $\mathcal{P}$ is either the Menger property or the property of Hurewicz.

Definition

We will say, that a topological space X is *Menger* ( ) if for every sequence $(\mathscr{U}_n)_{n\in\omega}$ of open covers of X, there is a sequence $(\mathscr{V}_n)_{n\in\omega}$ such that for every n, $\mathscr{V}_n$ is a finite subfamily of $\mathscr{U}_n$ and the family $\bigcup_{n\in\omega}\mathscr{V}_n$ covers X ( ).

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Proposition (Telgársky, 1984) (Kočinac, 2006)

A space X is Menger (Hurewicz resp.) if and only if X is Lindelöf and projectively Menger (projectively Hurewicz resp.).

Space in its compactification

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Theorem (Smirnov)

X is Lindelöf $\longleftrightarrow$ for every compact $K \subseteq bX \setminus X$ there exists a set $G \subseteq bX$ of type G_δ such that $K \subseteq G \subseteq bX \setminus X$.

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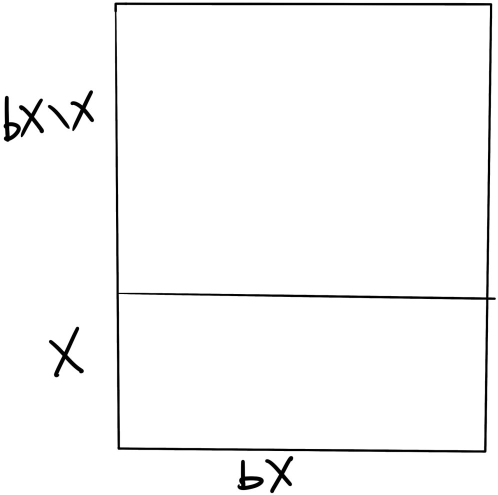
Theorem (F. Tall, 2011)

For a space X the following conditions are equivalent:

1. X has the Hurewicz property,
2. for every σ-compact subset F of the remainder $bX \setminus X$, there exists a G_δ-subset G of bX such that $F \subseteq G \subseteq bX \setminus X$.

TALL'S Thm.



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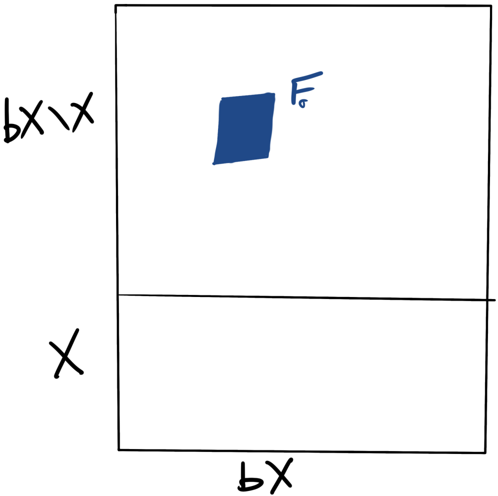

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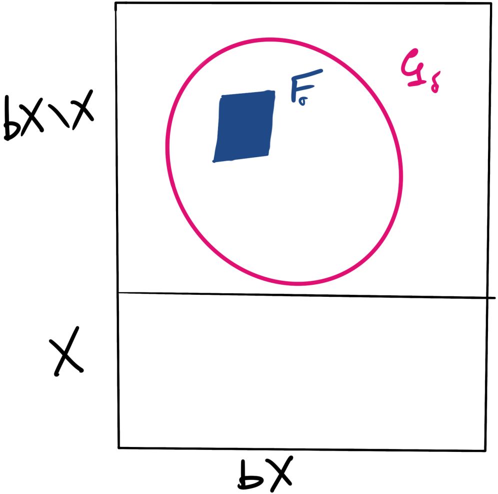

Characterization of projective Hurewicz property

Recall that a subset A of a topological space X is called a *zero-set* if there is a continuous map $f : X \to [0,1]$ such that $A = f^{-1}(0)$.

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The following conditions are equivalent:

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Proposition (Krupski, K.)

For a space X the following conditions are equivalent:

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2. for every $F \subseteq \beta X \setminus X$, such that F is a countable union of zero-sets in βX, there exists a G_δ subset G of βX with $F \subseteq G \subseteq \beta X \setminus X$.

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Projective Menger property also can be characterised in this way, but first we need to look closer at the notion of topological games.

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Player II wins the game if $\emptyset \neq \bigcap_{n\in\omega} U_n (= \bigcap_{n\in\omega} V_n) \subseteq X$.
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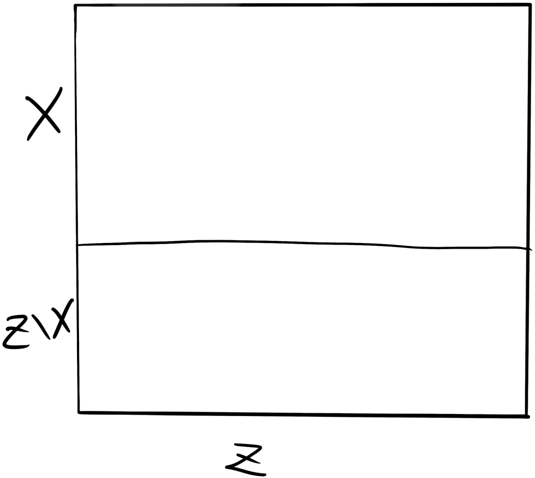

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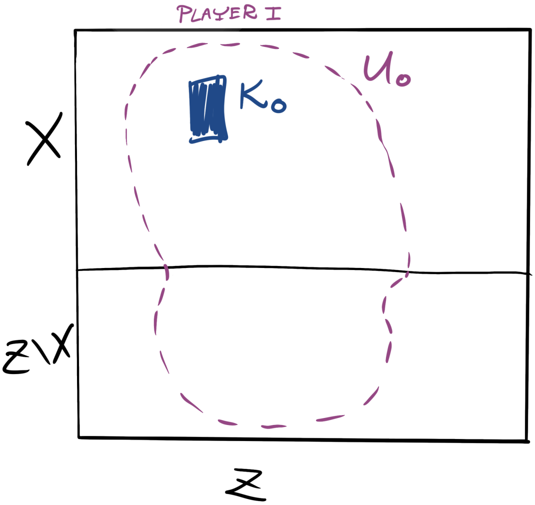

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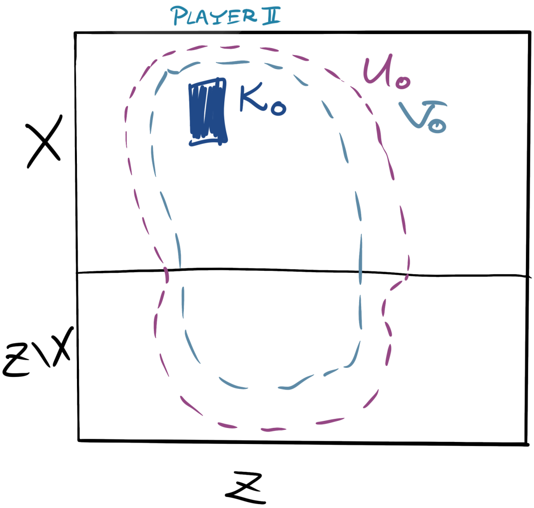

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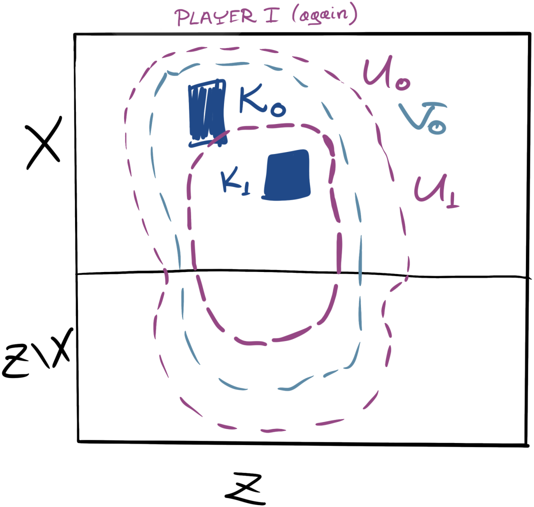

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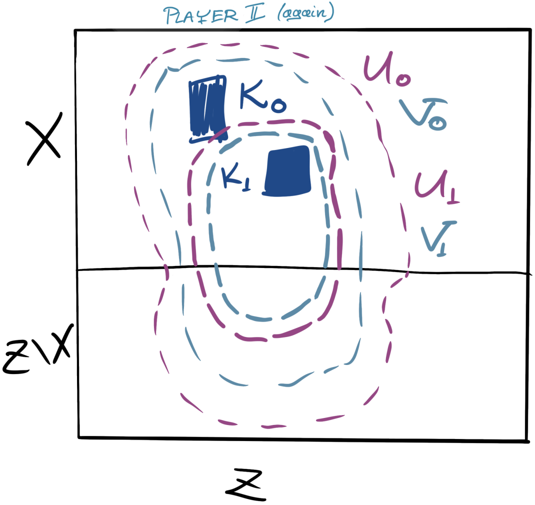

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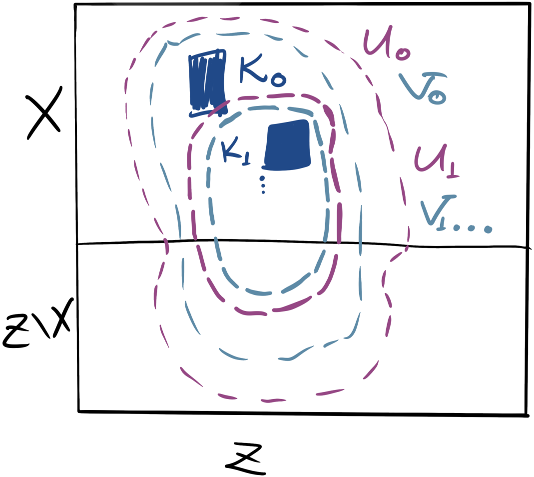
AND SO ON...



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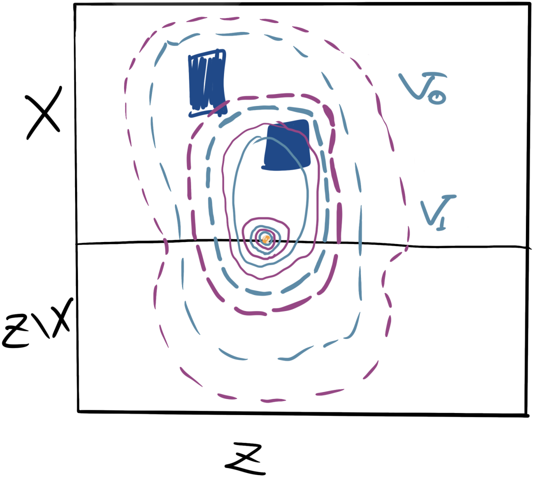

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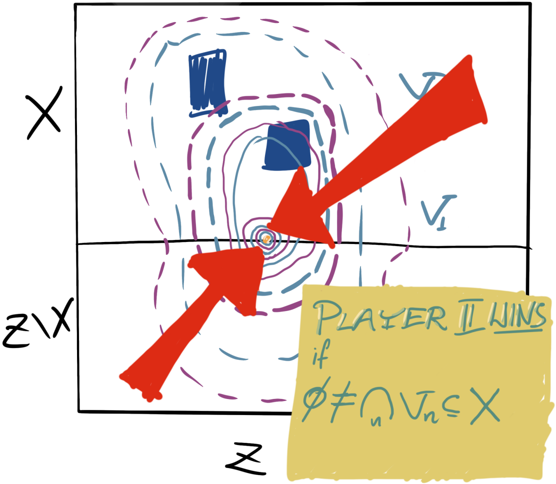

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2. For every sequence $(\mathscr{U}_n)_{n\in\omega}$ of countable covers of X by cozero-sets, there is a sequence $(\mathscr{V}_n)_{n\in\omega}$ such that for every n, $\mathscr{V}_n$ is a finite subfamily of $\mathscr{U}_n$ and the family $\bigcup_{n\in\omega}\mathscr{V}_n$ covers X.

Characterization of projective Menger property

Recall that a space X is *projectively Menger* provided every separable metrizable continuous image of X is Menger. The following result was established by Bonanzinga *et al.*

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Again, the above suggests the following counterpart to the theorem characterising property of Menger.

Characterization of projective Menger property (cont.)

Proposition (Krupski, K.)

For a space X the following conditions are equivalent:

1. X is projectively Menger,
2. Player I has no winning strategy in the z-Porada game $z\mathscr{P}(\beta X, \beta X \setminus X)$.

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- Assume X is projectively Menger and fix any strategy σ for Player I in the $z\mathscr{P}(\beta X, \beta X \setminus X)$ game.

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- After pulling back anwsers for Player II, one checks that startegy σ is not winning.

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